# Stability breakdown along a line of equilibria in nonlinear circuits with memristors

Ricardo Riaza

*Abstract*—The design in 2008 of a device with a memristive characteristic has had a great impact in electronics, specially at the nanometer scale. This device, whose existence was predicted by Leon Chua in 1971 for symmetry reasons, is governed in a flux-controlled setting by a relation of the form $i = W(\varphi)v$, and systematically leads to the presence of non-isolated equilibria. In this communication we examine how the stability of such manifolds of equilibria may break down when normal hyperbolicity fails. This phenomenon may be due to the transition of an eigenvalue either through the origin or through infinity. Our approach is a graph-theoretic one, aiming at the analysis of such phenomena in terms of the topology of the digraph underlying the circuit.

*Keywords*—Nonlinear circuit, memristor, equilibrium, stability, normal hyperbolicity, bifurcation.

## I. Introduction

MEMRISTORS are electronic devices defined by a charge-flux characteristic. Their existence was predicted for symmetry reasons by Leon Chua in 1971, since resistors, capacitors and inductors are defined by voltage-current, charge-voltage and flux-current relations, respectively. The charge-flux characteristic was the only one lacking in this set of relations, since the charge-current and the flux-voltage pairs are related by the electromagnetic laws $q' = i$, $\varphi' = v$. The design of such a *memory-resistor* or *memristor* at the nanometer scale announced by an HP team in 2008 [1] has driven a lot of attention to these devices. The flux-charge relation may have either a charge-controlled form $\varphi = \phi(q)$ or a flux-controlled one $q = \sigma(\varphi)$ [2]. We will focus on the latter but dual results apply to the charge-controlled case.

Applications of memristors and other memory devices are being reported in many fields: see [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17] and references therein. In particular, a significant impact in industry is expected to happen in the near future because of the use of memristors in non-volatile memory design. Not only from the point of view of applications but also from a mathematical perspective this device poses challenging problems. In this communication we focus on stability problems related to the systematic presence of manifolds of non-isolated equilibria in circuits including this device. These problems will be addressed in Section III, after compiling some introductory material on Section II. Finally, some concluding remarks can be found on Section II.

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## II. Memristive circuits

### A. The memristor

As indicated above, a flux-controlled memristor is defined by a nonlinear, differentiable relation

$$q = \sigma(\varphi);$$

time derivation yields, by means of the identities $q' = i$, $\varphi' = v$, the current-voltage characteristic

$$i = W(\varphi)v, \tag{1}$$

where $W(\varphi) = \sigma'(\varphi)$ is the *memductance*. The dual case is defined by a flux-charge relation $\varphi = \phi(q)$ which yields a voltage-current characteristic of the form

$$v = M(q)i, \tag{2}$$

where $M(q) = \phi'(q)$ is the so-called *memristance*. Note that (2) is reminiscent of Ohm's law, but the "resistance" $M(q)$ now depends on the charge $q$, which is the time-integral of the current $i$; for this reason the device' characteristic keeps track of its own history. The name *memristor*, which is an abbreviation of *memory-resistor*, comes from this remark [2]. Similar remarks apply to the flux-controlled case defined by (1); this form will be assumed throughout the document.

### B. Branch-oriented modelling of memristive circuits

For the sake of simplicity we will focus the attention on a restricted class of memristive circuits, just including flux-controlled memristors, voltage-controlled resistors, and capacitors. We will refer to these as WGC-circuits. Dual devices (charge-controlled memristors, current-controlled resistors, and inductors) as well as voltage and current sources are precluded in order to keep the discussion as simple as possible, but the results may be proved to hold in general. The essential mathematical aspects of the discussion are already present in WGC-circuits. To focus on cases with a one-dimensional manifold (that is, a line) of equilibria we will further assume that the circuit has a unique memristor.

Such circuits can be described by the differential-algebraic model (cf. [18], [19])

$$\begin{aligned} \varphi_m' &= v_m & (3)\\ C(v_c)v_c' &= i_c & (4)\\ 0 &= B_m v_m + B_c v_c + B_r v_r & (5)\\ 0 &= Q_m W(\varphi_m) v_m + Q_c i_c + Q_r g(v_r). & (6) \end{aligned}$$

Here the subscripts $m$, $c$, $r$ correspond to memristors, capacitors and resistors; $C(v_c)$ is the incremental capacitance matrix, and $i_r = g(v_r)$ is the current-voltage characteristic of resistors, which is assumed to be differentiable; the incremental conductance matrix is then $G(v_r) = g'(v_r)$. Note that (3)-(6) is a branch-oriented model (cf. [18]) which uses a description of Kirchhoff laws in the form $Bv = 0$, $Qi = 0$ in terms of reduced loop and cutset matrices $B$ and $Q$. The columns of these matrices are split according to the nature of the different devices, so that $B = (B_m \;\; B_c \;\; B_r)$, $Q = (Q_m \;\; Q_c \;\; Q_r)$ (find details in [18], [20]).

*Working scenario.* Both $C(v_c)$ and $G(v_r)$ are assumed to be positive definite everywhere; in circuit-theoretic terms, this means that capacitors and resistors are strictly locally passive. We also assume that $g(0) = 0$, and focus the analysis on the line of equilibria defined by the vanishing of the right-hand side of (3)-(6) when $v_m = i_c = v_c = v_r = 0$, in order to examine the qualitative behavior of the system as the variable $\varphi_m$ changes along this line. Specifically, we will assume that $W(0) = 0$ and $W'(0) \neq 0$, so that the memristor becomes active as $\varphi_m$ undergoes the null value. Recall that a memristor is said to be strictly locally passive (resp. active) at a given value of $\varphi$ if $W(\varphi) > 0$ (resp. $W(\varphi) < 0$).

The vanishing of $W$ may lead to the loss of normal hyperbolicity of the line of equilibria described above. An $m$-dimensional manifold of equilibrium points in an $n$-dimensional system is said to be *normally hyperbolic* if $n-m$ eigenvalues of the linearization are away from the imaginary axis [21]: note that $m$ eigenvalues necessarily vanish because of the presence of an $m$-dimensional manifold of equilibria. In our context, depending on the topology of the circuit and, specifically, on the location of the memristor, the vanishing of the memductance $W$ may result in the loss of normal hyperbolicity and different bifurcation phenomena may follow. Some of these phenomena are addressed in the main results reported in this communication, which can be found in Section III below.

## III. Stability breakdown

### A. Double zero eigenvalue

The linearization of a dynamical system along a line of equilibria obviously displays a null eigenvalue. When a second eigenvalue undergoes the origin, a *transcritical bifurcation without parameters* occurs generically [22], [23], [24]. If the remaining eigenvalues have negative real parts, this implies that the line of equilibria experiences a loss of stability in the region where the bifurcating eigenvalue becomes positive (find details in the references just cited). We discuss below certain circuit-theoretic conditions which characterize this phenomenon for the set of circuits presented above.

**Proposition 1.** *Consider the system (3)-(6) in the working scenario described above. If the circuit has a unique WC-cutset which actually includes the memristor, and there are neither C-loops nor C-cutsets, then the null eigenvalue of the linearization of (3)-(6) along the line of equilibria becomes a double one at the origin. This corresponds to a second eigenvalue which crosses the origin and becomes positive as $W$ becomes negative (that is, as the memristor becomes strictly locally active) when $\varphi$ varies. The remaining eigenvalues have negative real parts, and therefore this transition implies the loss of stability of the equilibrium line when $W$ becomes negative.*

Both the statement and the proof of this result make use of some notions and properties of digraph theory which we compile in what follows. Find detailed introductions to digraph theory and its use in circuit analysis in [18], [19], [20], [25], [26], [27] A *loop* or *cycle* in a directed graph (or digraph) is the set of branches in a closed path without self-intersections. A *cutset* $K$ is a set of branches whose removal increases the number of connected components of the digraph, and which is minimal with respect to this property, that is, the removal of any proper subset of $K$ does not increase the number of components. In a connected digraph, a cutset is just a minimal disconnecting set of branches. The removal of the branches of a cutset increases the number of connected components by exactly one. We assume that the digraph has neither bridges (cutsets defined by a single branch) nor selfloops (loops formed by a unique branch).

Given a set $K$ of branches, we will denote by $B_K$ (resp. $Q_K$) the submatrix of $B$ (resp. of $Q$) defined by the columns which correspond to $K$-branches. The absence of loops or cutsets including only $K$-devices can be easily characterized in terms of $B_K$ and $Q_K$; specifically, $K$ does not include cutsets if and only if $B_K$ has full column rank (i.e. $\ker B_K = \{0\}$) and, analogously, it does not include loops if and only if $Q_K$ has full column rank.

*Proof of proposition 1:* The linearization of (3)-(6) at a generic equilibrium is defined by the matrix pencil (cf. subsection III-B below) $\lambda A - J$, where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & C(0) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{7}$$

and $J$ is the matrix of partial derivatives of the right-hand side of (3)-(6) with respect to the variables $\varphi_m$, $v_c$, $v_m$, $i_c$, $v_r$, that is,

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_c & 0 \\ 0 & B_c & B_m & 0 & B_r \\ 0 & 0 & Q_m W(\varphi_m) & Q_c & Q_r G(0) \end{pmatrix}. \tag{8}$$

One can easily check that $\det(\lambda A - J)$ reads as

$$\det \begin{pmatrix} \lambda & 0 & -1 & 0 & 0 \\ 0 & \lambda C(0) & 0 & -I_c & 0 \\ 0 & -B_c & -B_m & 0 & -B_r \\ 0 & 0 & -Q_m W(\varphi_m) & -Q_c & -Q_r G(0) \end{pmatrix}, \tag{9}$$

and, for $\varphi = 0$,

$$\det \begin{pmatrix} \lambda & 0 & -1 & 0 & 0 \\ 0 & \lambda C(0) & 0 & -I_c & 0 \\ 0 & -B_c & -B_m & 0 & -B_r \\ 0 & 0 & 0 & -Q_c & -Q_r G(0) \end{pmatrix},$$

since $W(0) = 0$ because of the working assumptions. In this case, $\lambda = 0$ is indeed a double zero eigenvalue because of the fact that

$$\begin{pmatrix} B_c & B_m & B_r \\ 0 & 0 & Q_r G(0) \end{pmatrix} \tag{10}$$

is a singular matrix with a minimal rank deficiency: this is a consequence of the existence of a unique WC-cutset, which makes the kernel of $(B_c\ B_m)$ non-trivial and, actually, one-dimensional. The positive definiteness of the conductance matrix $G(0)$ transfers this minimal rank deficiency to the matrix (10) and this implies that the null eigenvalue is indeed a double one when $\varphi = 0$.

The fact that this second null eigenvalue actually crosses the origin as $\varphi$ varies follows from the characterization of the transcritical bifurcation without parameters reported in [22], [23], [24]. Skipping technical details for the sake of brevity, this is specifically a consequence of the assumption $W'(0) \neq 0$; note that, together with $W(0) = 0$, this yields a sign change in $W(\varphi)$ as $\varphi$ undergoes the null value. Owing to the results in [28], for positive values of $W$ (recall that both $G(0)$ and $C(0)$ are positive definite) all non-vanishing eigenvalues have non-positive real parts, one real eigenvalue actually becoming positive as $W$ takes on negative values. Finally, the fact that the remaining eigenvalues are away from the imaginary axis follows from the results discussed in [29], according to which the absence of inductors is enough to guarantee, under the assumed absence of C-loops and C-cutsets, that no purely imaginary eigenvalues are depicted in the linearized problem. ■

*Example 1.* Proposition 1 above provides a circuit-theoretic description of the topological reasons supporting the stability loss example discussed in [30]. Indeed, the simplest instance of a circuit verifying the assumptions in Proposition 1 is depicted in Figure 1.

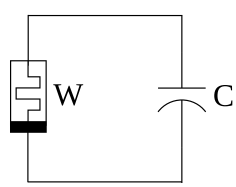


Fig. 1. Example 1

Assuming that the capacitor is a linear one, with positive capacitance $C$, the circuit equations amount to (cf. [30])

$$\begin{aligned} \varphi_m' &= v \\ Cv' &= -W(\varphi_m)v. \end{aligned}$$

It is a simple matter to check that the equilibrium line, defined by $v = 0$ and parameterized by $\varphi_m$, becomes unstable as $W$ becomes negative. The two eigenvalues can be checked to read, at a generic equilibrium,

$$\lambda_1 = 0,\ \ \lambda_2 = -\frac{W(\varphi_m)}{C},$$

and, assuming $W(0) = 0$, $W'(0) \neq 0$, we have at the origin a double zero eigenvalue with geometric multiplicity one which is indeed responsible for the stability breakdown; note that, indeed, the second eigenvalue becomes positive as $W$ takes on negative values.

In circuit-theoretic terms, this is just a consequence of the fact that the two branches of the circuit define a WC-cutset; together with the absence of C-loops and C-cutsets, this means that Proposition 1 applies.

### B. Eigenvalue divergence through $\pm\infty$

It is interesting to note that the dual behavior to the one above may be depicted by divergence of one eigenvalue of the pencil $\lambda A - J$, with $A$ and $J$ given in (7) and (8). Given two matrices $A,\ B$ in $\mathbb{R}^{n\times n}$ the *matrix pencil* $\{A, B\}$ is defined as the one-parameter family $\lambda A + B$. If the polynomial (in $\lambda$) $\det(\lambda A + B)$ does not vanish identically (that is, if there exists at least one value of $\lambda$ for which $\det(\lambda A + B) \neq 0$), the matrix pencil is called *regular*. The (finite) eigenvalues of a regular matrix pencil $\{A, B\}$ are the values of $\lambda \in \mathbb{C}$ for which $\det(\lambda A + B) = 0$. The polynomial $\det(\lambda A + B)$ of a regular pencil has in general a degree $m \leq n$, with $m < n$ when $A$ is a singular matrix; in the latter case case we say that the pencil has $n - m$ infinite eigenvalues.

In our setting, provided that the derivative of the right-hand side of (3)-(6) with respect to the variables $v_m$, $i_c$, $v_r$ is non-singular, then the pencil $\lambda A - J$ has exactly $m$ eigenvalues, where $m$ is the total number of memristors and capacitors, because of the index-one nature of the differential-algebraic equations modelling the circuit [18], [19]. By contrast, the vanishing of $W(\varphi)$ at a given value of $\varphi$ may result in the appearance of additional infinite eigenvalues and, again, in a stability breakdown along the line of equilibria; this can be seen as a result of the index jump resulting from the singularity (cf. [30], [31], [32]).

We illustrate this behavior by means of a simple example, obtained after replacing the capacitor in Figure 1 by an inductor, as depicted in Figure 2. The key idea is that, open-circuiting the memristor, the circuit results in an L-cutset, which yields an index-two circuit configuration.

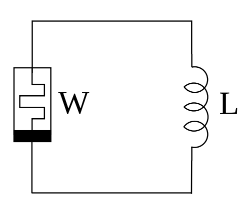


Fig. 2. Example 2

The circuit equations now read as

$$\varphi_m' = v \tag{11}$$
$$Li' = -v \tag{12}$$
$$0 = i - W(\varphi_m)v. \tag{13}$$

Assume $L > 0$. One eigenvalue of (11)-(13) is fixed at the origin, consistently with the existence of the branch of equilibria defined by the identities $i = v = 0$. The second eigenvalue can be checked to read as

$$\frac{-1}{LW(\varphi_m)}$$

and jumps from $-\infty$ to $+\infty$ as $W$ crosses zero and becomes negative. Note that, again, the change of stability occurs along the line of equilibria.

A topological characterization of this phenomenon in memristive circuits, in analogous terms to the ones of Proposition 1, is the object of undergoing research.

## IV. Conclusion

Many mathematical properties of memristive circuits remain to be solved. Some chaotic phenomena have been explored in [33], [34], [35], but a complete analysis of the analytical properties of manifolds of equilibria in problems with one or several memristors has not yet been fully addressed in the literature. Such results should be relevant in practical applications involving memristors and other mem-devices.

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## References


[1] D. B. Strukov, G. S. Snider, D. R. Stewart and R. S. Williams, The missing memristor found, *Nature* **453** (2008) 80-83.

[2] L. O. Chua, Memristor – The missing circuit element, *IEEE Trans. Circuit Theory* **18** (1971) 507-519.

[3] A. Ascoli, T. Schmidt, R. Tetzlaff and F. Corinto, Application of the Volterra series paradigm to memristive systems, in R. Tetzlaff (ed.), *Memristors and Memristive Systems,* pp. 163-191, Springer, 2014.

[4] D. Biolek, Z. Biolek and V. Biolkova, SPICE modeling of memristive, memcapacitive and meminductive systems, *Proc. Eur. Conf. Circuit Theor. Design 2009*, pp. 249-252, 2009.

[5] F. Corinto, A. Ascoli and M. Gilli, Analysis of current-voltage characteristics for memristive elements in pattern recognition systems, *Internat. J. Circuit Theory Appl.* **40** (2012) 1277-1320.

[6] M. Di Ventra, Y. V. Pershin and L. O. Chua, Circuit elements with memory: memristors, memcapacitors and meminductors, *Proc. IEEE* **97** (2009) 1717-1724.

[7] M. Itoh and L. O. Chua, Memristor oscillators, *Internat. J. Bifur. Chaos* **18** (2008) 3183-3206.

[8] M. Itoh and L. O. Chua, Memristor Hamiltonian circuits, *Internat. J. Bifur. Chaos* **21** (2011) 2395-2425.

[9] L. Jansen, M. Matthes and C. Tischendorf, Global unique solvability for memristive circuit DAEs of index 1, *Int. J. Circuit Theory Appl.*, in press, 2014.

[10] D. Jeltsema and A. J. van der Schaft, Memristive port-Hamiltonian systems, *Math. Comp. Model. Dyn. Sys.* **16** (2010) 75-93.

[11] D. Jeltsema and A. Doria-Cerezo, Port-Hamiltonian formulation of systems with memory, *Proc. IEEE* **100** (2012) 1928-1937.

[12] O. Kavehei, A. Iqbal, Y. S. Kim, K. Eshraghian, S. F. Al-Sarawi and D. Abbott, The fourth element: characteristics, modelling and electromagnetic theory of the memristor, *Proc. R. Soc. A* **466** (2010) 2175-2202.

[13] M. Messias, C. Nespoli and V. A. Botta, Hopf bifurcation from lines of equilibria without parameters in memristors oscillators, *Internat. J. Bifur. Chaos* **20** (2010) 437-450.

[14] Y. V. Pershin and M. Di Ventra, Memory effects in complex materials and nanoscale systems, *Advances in Physics* **60** (2011) 145-227.

[15] Y. V. Pershin and M. Di Ventra, Neuromorphic, digital and quantum computation with memory circuit elements, *Proc. IEEE* **100** (2012) 2071-2080.

[16] R. Riaza, Nondegeneracy conditions for active memristive circuits, *IEEE Trans. Circuits and Systems - II* **57** (2010) 223-227.

[17] R. Tetzlaff (ed.), *Memristors and Memristive Systems,* Springer, 2014.

[18] R. Riaza, *Differential-Algebraic Systems,* World Scientific, 2008.

[19] C. Tischendorf, Coupled systems of differential algebraic and partial differential equations in circuit and device simulation. Modeling and numerical analysis, Habilitationsschrift, Inst. Math., Humboldt University, Berlin, 2003.

[20] L. O. Chua, C. A. Desoer and E. S. Kuh, *Linear and Nonlinear Circuits*, McGraw-Hill, 1987.

[21] B. Aulbach, *Continuous and Discrete Dynamics near Manifolds of Equilibria*, Lect. Note Math. 1058, Springer-Verlag, 1984.

[22] B. Fiedler, S. Liebscher, and J. C. Alexander, Generic Hopf bifurcation from lines of equilibria without parameters: I. Theory, *J. Differential Equations* **167** (2000) 16-35.

[23] B. Fiedler and S. Liebscher, Generic Hopf bifurcation from lines of equilibria without parameters: II. Systems of viscous hyperbolic balance laws, *SIAM J. Math. Anal.* **31** (2000) 1396-1404.

[24] S. Liebscher, *Bifurcation without Parameters,* Springer, 2015.

[25] B. Andrásfai, *Introductory Graph Theory,* Akadémiai Kiadó, Budapest, 1977.

[26] B. Andrásfai, *Graph Theory: Flows, Matrices,* Adam Hilger, 1991.

[27] B. Bollobás, *Modern Graph Theory*, Springer-Verlag, 1998.

[28] R. Riaza and C. Tischendorf, Qualitative features of matrix pencils and DAEs arising in circuit dynamics, *Dynamical Systems* **22** (2007) 107-131.

[29] R. Riaza and C. Tischendorf, The hyperbolicity problem in electrical circuit theory, *Math. Methods Appl. Sci.* **33** (2010) 2037-2049.

[30] R. Riaza, Manifolds of equilibria and bifurcations without parameters in memristive circuits, *SIAM J. Appl. Math.* **72** (2012) 877-896.

[31] R. E. Beardmore, The singularity-induced bifurcation and its Kronecker normal form, *SIAM J. Matrix Anal. Appl.* **23** (2001) 126-137.

[32] V. Venkatasubramanian, H. Schättler and J. Zaborszky, Local bifurcations and feasibility regions in differential-algebraic systems, *IEEE Trans. Aut. Contr.* **40** (1995) 1992-2013.

[33] B. Bao, Z. Ma, J. Xu, Z. Liu and Q. Xu, A simple memristor chaotic circuit with complex dynamics, *Internat. J. Bifurcation and Chaos* **21** (2011) 2629-2645.

[34] B. Muthuswamy, Implementing memristor based chaotic circuits, *Internat. J. Bifur. Chaos* **20** (2010) 1335-1350.

[35] B. Muthuswamy and L. O. Chua, Simplest chaotic circuit, *Internat. J. Bifur. Chaos* **20** (2010) 1567-1580.


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